Proceedings of the 14th International Conference of the Croatian Nuclear Society
Zadar, Croatia, June 9 – 12, 2024 *Paper No. 122*

Algebraic Decision Diagrams in Multi-State Reliability Analysis of Nuclear Power Plant Systems

Reni Banov
Zagreb University of Applied Sciences
Vrbik 8, Zagreb, Croatia
reni.banov@tvz.hr

ABSTRACT

The reliability analysis which takes in account multiple possible states of components is considered as a multi-state reliability analysis. Although multi-state reliability models provide more realistic and more precise representations of Nuclear Power Plant systems, their analysis is considered far more complex and demanding. A popular way to approach such an analysis consists in the application of stochastic processes, especially Markov chains. Unfortunately, the number of transitional state spaces in Markov chains grows exponentially with the increase in the number of component states. Algebraic Decision Diagrams prove to be an efficient data structure for storing Markov chains and can be used to perform such an analysis. Preliminary results on smaller models show the applicability of algebraic decision diagrams to represent Markov chains describing hundreds or even thousands of states from Nuclear Power Plant systems with multi-state components. These results justify the implementation of algebraic decision diagrams techniques to automate Markov chain analysis of systems with extremely large state space models. Experimental results from a prototype of an algebraic decision diagrams-based analysis of a simplified model of the Nuclear Component Cooling Water system are provided to substantiate conclusions.

Keywords: *Probabilistic Safety Assessment (PSA), Dynamic Fault Tree (DFT), Stochastic Process (SP), Algebraic Decision Diagrams (ADD), Multi-state Systems (MSS)*

1 INTRODUCTION

In recent times, modern engineering faces problems concerning the modelling of the dynamic behaviour of processes, i.e., time-varying system states and the rules according to which such changes occur. The development of such models that can describe dynamic changes in systems depends to a significant extent on the application of the mathematical theory of stochastic processes. For this purpose, the application of Markov chains for conducting analysis and for creating models for determining reliability models based on the probabilities of occurrence of possible states proves to be adequate. The very procedure of analysing such systems with components that can have three or more states is called the multi-state reliability analysis. Nuclear engineering is not excepted from the multi-state reliability analysis of dynamic changes in systems. Moreover, because of the exceptional importance (and sensitivity) of nuclear energy such analyses represent an important item in the assessment of the reliability and safety of critical systems in nuclear power plants [1,2]. This paper aims at illustrating the analysis of dynamic changes of a simple water-cooling system in a nuclear power plant.

The Nuclear Component Cooling Water (NCCW) system is used to remove excess heat from components that may contain potentially radioactive materials. However, aqueous heat removal creates a potentially radioactive fluid wherefore it is extremely important to provide a reliable environment. Authors Xing et al. [3] compared the fault tree model against the Markov model to determine the frequency of the initiating cooling loss event for both power and shutdown conditions. Their conclusions are that for systems with short mission time and failure mode dominated by independent failure the effect of repair is significant. According to the authors the Reliability Block Diagram (RBD) of the NCCW system contains two parallel pump trains connected in series with a heat exchanger train as shown in Figure 1.

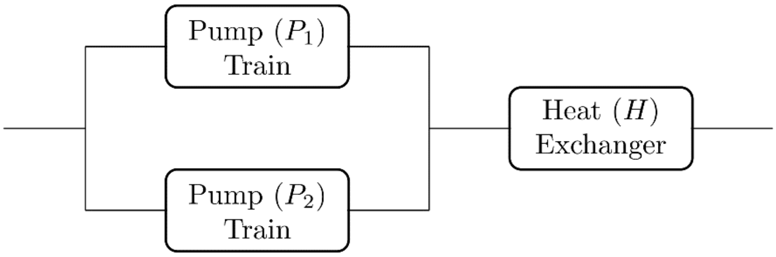

Figure 1: NCCW Reliability Block Diagram

While both pump trains are repairable, the heat exchanger is a non-repairable component. This configuration makes the Markov chain model the fitting choice to perform a system reliability analysis.

2 PRELIMINARIES

2.1 Continuous Markov Chains (CMC)

At the end of the nineteenth century, the mathematician Markov analysed processes in which the value of the random variable is determined only based on the preceding value, in other words, the current state of a random process does not depend on the path but only on the previous state, the so-called forgetting property. His considerations led to the development of the theory of stochastic processes and processes for which the forgetting property is valid. Today, such processes are called Markov processes, named in his honour.

Formally, at some time *t* we denote the state of the system (or process) by the state vector ***X(t)***, where the elements of the vector are random variables indicating the functional status (for example, Normal/Failure) for each component in the system. If the system consists of *n* components, then the random vector ***X(t)*** can take at most 2^n different combinations of discrete values, i.e., it can be in one of the 2^n possible states. Usually, for the system with *n* components possible states are denoted by numbers from the set $S = \{0,1,2,\dots,2^n - 1\}$, and $P_i(t)$ will denote the probability that the system is in state $s_i \in S$ at a particular moment *t*, i.e., the following definition holds $P_i(t) = P(\boldsymbol{X}(t) = s_i)$. Having those definitions, the forgetting property can be written as

$$P(\boldsymbol{X}(t_n) = s_n \mid s_{n-1},\dots,s_0) = P(\boldsymbol{X}(t_n) = s_n \mid \boldsymbol{X}(t_{n-1}) = s_{n-1}) \qquad (1)$$

where $0 \leq t_0 < t_1 < \cdots < t_{n-1} < t_n$ is a strictly increasing sequence of moments in time. A continuous stochastic process for which the forgetting property holds is denoted as the Markov chain. In addition, we imply the independence of the Markov chain on real time, but only on time intervals for the transition to a new state, in other words, for any choice of states $s_i, s_j \in S$ and the time interval $\tau > 0$, thus for any moment in time the following statement is valid

$$P(\boldsymbol{X}(t+\tau) = s_j \mid \boldsymbol{X}(t) = s_i) = P(\boldsymbol{X}(\tau) = s_j \mid \boldsymbol{X}(0) = s_i) = P_{ij}(\tau) \qquad (2)$$

A stochastic process with this additional property is called the homogeneous Markov chain or sometimes the process with stationary transition probabilities P_{ij}. Therefore, transition probabilities are written by using the matrix

$$\boldsymbol{P}(\tau) = \begin{bmatrix} P_{00}(\tau) & \cdots & P_{0n}(\tau) \\ \vdots & \ddots & \vdots \\ P_{n0}(\tau) & \cdots & P_{nn}(\tau) \end{bmatrix} \tag{3}$$

with an additional row normalisation condition $\sum_{j=0}^{n} P_{ij}(\tau) = 1$, i.e., the process from the state s_i in the moment *t*, at any subsequent moment will pass into one of the possible states (including state s_i). We are interested in the distribution of the retention time in some state of Markov chains. Without reducing the generality let us assume that the Markov chain for a moment *t* is in state s_i. Since the process fulfils a Markov property we can conclude that the probability of staying in the same state will at some point be determined only by the probability of transition to the same state, i.e., it is determined by the constant probability of $P_{ii}(0)$. We can say that from the forgetting property of a homogeneous Markov process an exponential random distribution of the retention time follows. Also, we can conclude from the Markov property that exponentially distributed random variables describing the retention time are independent of each other. Defined in that way, the homogeneous Markov process can be represented by the Algebraic Decision Diagrams (ADD) as shown in the paper [4].

2.2 Algebraic Decision Diagrams (ADD)

The Binary Decision Diagrams (BDD) structure was introduced by Akers [5] and gained true popularity with Bryant's seminal work [6]. It is distinguished for a compact representation of Boolean functions and for an efficient manipulation with then. The BDD stands as a variant of the *directed acyclic graph* with two terminal vertices and efficiently represents a *truth table* of Boolean functions by encoding their values as paths from the top vertex to terminal vertices. Many variations of BDDs are developed for different purposes, for example, the Zero Decision Diagrams (ZDD) to represent combinatorial sets. Based on these works, a new structure of decision diagrams has been developed to represent functions that take on values in a set of real numbers, i.e., functions taking real values on Boolean vectors $f: B^n \rightarrow R$. The idea of Algebraic Decision Diagrams (ADD) for representing such functions was developed by Bahar et al. [7] with the intention of effectively implementing algorithms from numerical linear algebra. The ADD diagrams share a number of useful properties with their relatives BDDs, for example, canonical representation of functions, two ADDs describe the same function if and only if they are the same ADDs. In addition, they find their purpose in a number of applications: computations with large matrices including probability calculations for stationary states in Markov chains, probability verifications, planning for Artificial Intelligence (AI), and many other purposes.

Briefly, the ADD diagrams can be described by using a directed acyclic graph structure made of two types of vertices:

a) terminal vertices representing real constants for the values of the function,
b) and all other vertices represent the *If-Then-Else (ITE)* construction defined with the Shannon decomposition of function based on a value of the Boolean variable x_i

$$f(x_1, \dots, x_i, \dots, x_n) = \left(x_i \wedge f(x_1, \dots, 1_i, \dots, x_n)\right) \vee \left(\overline{x_i} \wedge f(x_1, \dots, 0_i, \dots, x_n)\right) \tag{4}$$

By comparing the ADD graph with the BDD structure we notice the basic difference between these two structures, namely; the ADD graph as opposed to the BDD graph structure can have more than two terminal vertexes. With this in mind we can consider the ADD graph as a natural structure for the symbolic representation of sparse square matrices.

3 APPLICATION OF ADD TO CMC MODEL CALCULATION

As shown in Figure 1, the NCCW system is built as a serial connection of two parallelly connected water pumps to a heat exchanger. Both pumps are repairable components, while the heat exchanger is a non-repairable component, i.e., the failure of the heat exchanger means the termination of functioning of the NCCW system. If we assume the exponential distribution during the retention of a component in a state, then systems containing repairable and non-repairable components can be analyzed as shown in the paper [8]. However, if the distribution of the transition time between states is not exponential but depends on the time of use of the component (e.g., Weibull's) then the system can be analyzed with discrete-time semi-Markov chains that are a generalization of Markov [9]. The technique of calculation applying ADD diagrams used in this paper is also applicable to models described by semi-Markov chains.

3.1 CMC model of NCCW system

Let us analyze the NCCW system assuming there is no common cause of failure for the two pumps, although such an analysis is not significantly more complex than the one shown herein, it has been dropped herein due to the simplicity of record. Let us look at the NCCW system for three different states as shown in the Markov chain model from Figure 2.

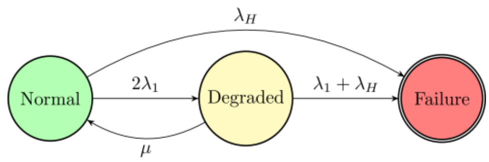

Figure 2: Markov model for the NCCW system

The Markov model shows transitions between the three states of the NCCW system:

1. A fully functional "**2P+H**" state in which both pumps and the heat exchanger are operational.
2. Degraded state "**P+H**" in which the system is still functional, but one of the pumps is malfunctioning.
3. System failure state when the system is not functional due to a failure on the heat exchanger or both pumps are malfunctioning. The failure state is the absorbing state of the Markov chain, i.e., the system remains in that state if it comes into it.

Kolmogorov's system of differential equations for solving the NCCW CMC model can be written based on the elaboration of the theoretical model as described in the book from Lisnianski and Levitin [10]. Let $p_N(t), p_D(t), p_F(t)$ represent the probability of system being in states "Normal", "Degraded", and "Failure" respectively, then the system of differential equations becomes

$$\begin{aligned} \frac{dp_F}{dt}(t) &= (\lambda_1 + \lambda_H)\, p_D(t) + \lambda_H\, p_N(t) \\ \frac{dp_D}{dt}(t) &= 2\lambda_1\, p_N(t) - \mu\, p_D(t) \\ \frac{dp_N}{dt}(t) &= \mu\, p_D(t) - (2\lambda_1 + \lambda_H)\, p_N(t) \end{aligned} \quad (5)$$

If we assume the initial normal state of the NCCW system, then the system of differential equations (5) can be solved in the same way as described in the paper [8] with the vector of the initial conditions $[\, p_F(0) = p_D(0) = 0,\ p_N(0) = 1]$. As can be seen from the Markov model, the NCCW system has three basic states. In the normal "2P+H" state all three components are fully operational. From this state the system can move to a degraded state ("P+H") when one of the pumps fail (λ_1 is failure rate for one pump), while the time to return to normal state is determined by the pump repair rate (parameter μ). Failure of the heat exchanger (λ_H is failure rate for heat exchanger) or failure of another pump brings the system into a non-functional state. Assuming a binary (Operational/Non-operational) state for each component the state space of the NCCW system consists of $2^3 = 8$ possible states as shown on the state space diagram in Figure 3.

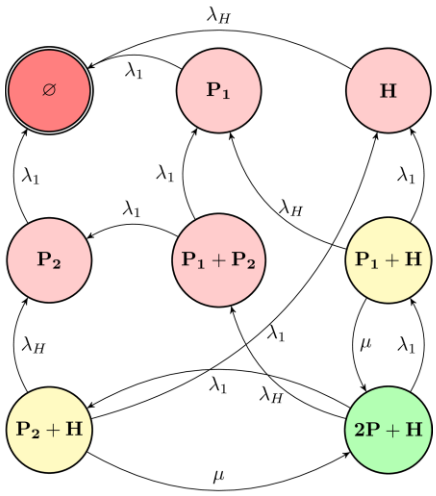

Figure 3: NCCW State Space Diagram

Let the state space of the NCCW system be ordered (indexed) as follows

$$S = \{\emptyset, P_1, P_2, P_1 + P_2, H, P_1 + H, P_2 + H, 2P + H\}$$

then s_0 is "Failure" state $(\emptyset)$, s_1 is P_1, up to the last state s_7 which is "Normal" $(2P + H)$ operational state. The transition rate matrix (Table 1) associated with this state space diagram is a square matrix of dimension eight as listed in Table 1.

Table 1: NCCW Transition rate matrix

Current state $(r_3 r_2 r_1)$	Destination state $(c_3 c_2 c_1)$							
	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_0	0	0	0	0	0	0	0	0
s_1	λ_1	0	0	0	0	0	0	0
s_2	λ_1	0	0	0	0	0	0	0
s_3	0	λ_1	λ_1	0	0	0	0	0
s_4	λ_H	0	0	0	0	0	0	0
s_5	0	λ_H	0	0	λ_1	0	0	μ
s_6	0	0	λ_H	0	λ_1	0	0	μ
s_7	0	0	0	λ_H	0	λ_1	λ_1	0

3.2 ADD representation of the CMC model

Let us introduce logical variables (r_1, r_2, r_3) for the index of the current state of the NCCW system and logical variables (c_1, c_2, c_3) for the index of the state into which the system transits after the time interval τ, then the ADD diagram of the transition rate matrix is shown in Figure 4.

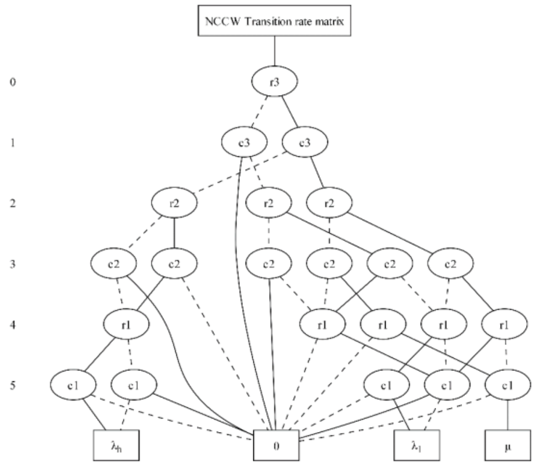

Figure 4: ADD diagram of the transition rate matrix

On the ADD diagram, for each node except the terminal nodes we notice two types of output branches (lines). A full line indicates a branch that corresponds to the restriction $f(x_1, \dots, 1_i, \dots, x_n)$, while a dashed line indicates a branch for restriction $f(x_1, \dots, 0_i, \dots, x_n)$ of the function on variable x_i in Shannon decomposition (4). Each path from the top node to terminal nodes corresponds to a single transition between the states of the NCCW system. For example, a path $\{r_3 = 1, c_3 = 1, r_2 = 0, c_2 = 0, r_1 = 1, c_1 = 0\}$ that ends in a terminal node λ_1 corresponds to the transition of the system from the state s_5 to the state s_4.

Normally, in practical application it will be necessary to determine the behaviour of the system at discrete time intervals, for example, at intervals of one hour, for a period of one month, after a year, etc. At discrete time intervals a continuous Markov chain can be approximated by the values of the discrete Markov chain. Namely, it can simply be shown that for continuous Markov chains by homogenous discretization of time we get a discrete Markov chain. Accordingly, assuming the exponential distribution for the retention time we can get a matrix of stationary transition probabilities for a fixed time interval ($\tau > 0$) from the transition rate matrix by simple procedure. Since we assumed that the retention time in some state is an exponentially distributed random variable $X \sim Exp(\lambda)$ with a transition rate λ equal to one of $(\lambda_1, \lambda_H, \mu)$ we can replace values of the terminal vertices in the ADD diagram with value

$$P(X \leq \tau) = 1 - e^{-\lambda\tau} \tag{6}$$

which corresponds to the probability of a transition between states in the time interval τ. Normalization of the matrix is achieved by placing diagonal elements in the newly created matrix $\boldsymbol{P}$ on values,

$$p_{ii} = 1 - \sum_{j \neq i}^{n} p_{ij} \tag{7}$$

i.e., on the probability of remaining the system in state s_i. In this way, the formed matrix $\boldsymbol{P}$ of the stationary transition probabilities allows the application of the power method (see [4],[11]) to calculate the probability of finding the NCCW system after $n\tau$ discrete time intervals. For example, for parameter values ($\lambda_1 = 7.7243 \cdot 10^{-6}$, $\lambda_H = 3.4243 \cdot 10^{-6}$, $\mu = 19.4^{-1}$) per hours, the ADD diagram of the $\boldsymbol{P}^T$ (transposed $\boldsymbol{P}$) matrix from the power method (see [4]) for the NCCW system model looks like in Figure 5.

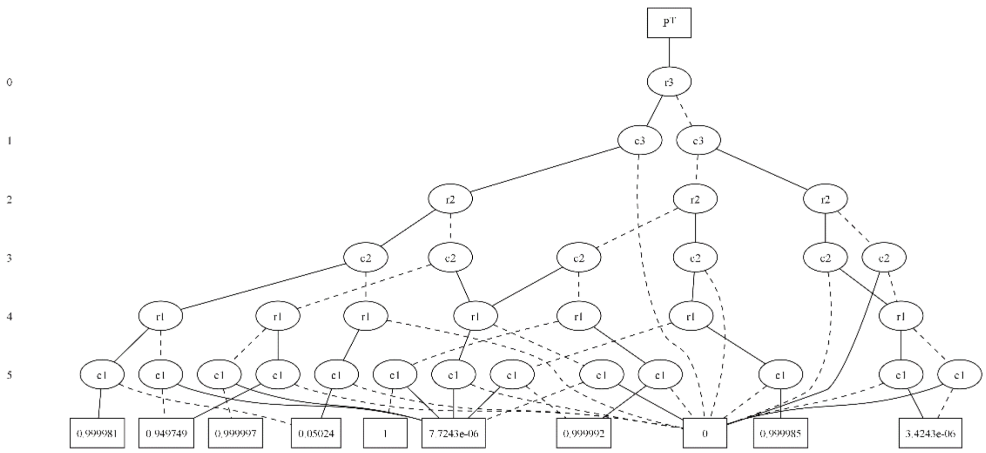

Figure 5 ADD diagram of transposed matrix $\boldsymbol{P}^T$ from the power method

where similarly to the transition rate matrix ADD representation, the paths from the top node to terminal nodes determines the probability of transition between two states.

4 DISCUSSION OF RESULTS

Let us apply the power method to the discretized CMC model of the NCCW system for the values of parameters that formed the transposed matrix (Figure 5.) of transition probabilities and compare the obtained results with the calculation of the CMC model using differential equations (5). All numerical calculations are caried out from the initial state "Normal" in which all three components are fully functional. The calculation of discrete numerical values by the power method was carried out using the implementation of the CUDD [12] software package for working with ADD diagrams. In the first group of tests numerically calculated probabilities of finding the NCCW system in one of the states from the state space S were compared for each hour over a period of one year. As can be seen in Figure 6. there are no significant discrepancies between numerically calculated and exact values of probabilities for the states of the CMC Markov model of the NCCW system as in Figure 2.

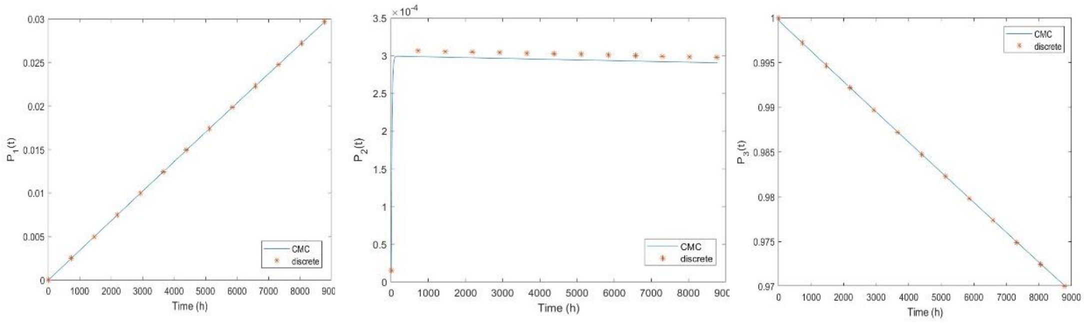

Figure 6: CMC Numerical vs exact computation (per hour) for one-year period

The only noticeable difference is found for "Degraded" state, which is a consequence of the normalization step (the sum of all probabilities must be 1) of the state vector in the power method. We notice the difference because the probability of being in state "Degraded" is the lowest, therefore, by normalizing the corrected probability differs by a small absolute value, amounting to less than 0,3 percent of relative error. By comparing the results for a larger time interval, for example ten years with a discretization of one year ($\tau = 1 year$), as shown in Figure 7, a slightly larger difference (less than 1% of relative error) is noticeable in the values for the probability of finding the NCCW system in one of the three states from the CMC model. The observed error is a consequence of rough discretization of the CMC model, namely, the time interval of one year is the main cause of the occurrence of error in the result.

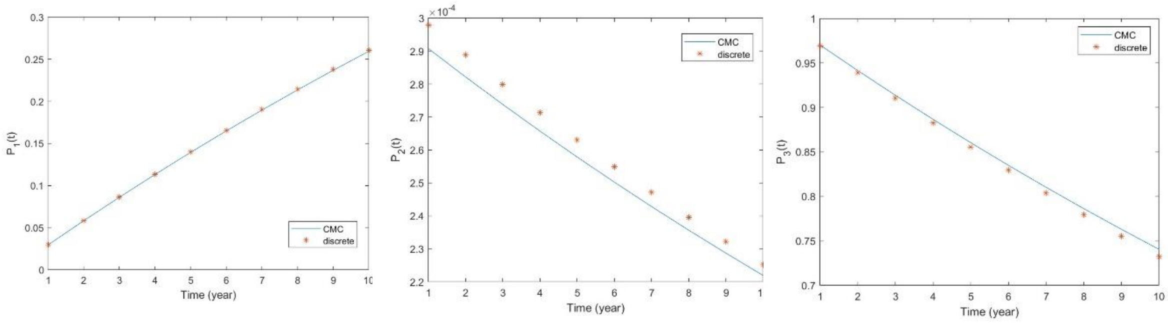

Figure 7: CMC Numerical vs exact computation (per year) for a ten-year period

It is interesting to bring forth how many nodes of ADD diagrams are needed to carry out the calculation of the discretized Markov chain. For example, in order to implement the calculation for a one year period with a discretization of one hour it took 175305 ADD nodes (~5MB) to save all intermediate results for matrices and state vectors. While for the implementation of the calculation for a ten-year period with the same discretization of one hour including the saving of all results 1753185 nodes (~50MB) were needed, therefore, a linear increase in the number of nodes is present. The time duration of the calculation is practically negligible, for example, for a one-year test it takes 0.08 seconds, while for ten-years it lasts for 1.58 seconds on the 12th generation Intel Core i5 CPU.

5 CONCLUSIONS AND FUTURE WORK

Multi-state probabilistic analyses can be of exceptional benefit for applications in nuclear energetic systems. In particular, the application of continuous Markov chains for the analysis of system dynamics, i.e., to analyse the behaviour of the system over time. It is undoubtedly the preferred way to calculate continuous Markov chains via Kolmogorov differential equations, since it is a way of acquiring analytical solutions for the behaviour of the system through time. However, with a large number of states the analytical procedure becomes demanding due to the size of the system (number) of differential equations. In such cases the application of discretized Markov chains for the implementation of approximate calculations appears to be suitable. The calculation for discrete Markov chains is usually carried out with sparse matrices for which the application of ADD diagrams is appropriate. As shown in this paper stochastic matrices that occur in the analysis of the discrete Markov chain can be effectively written in form of ADD diagrams, on which record the necessary matrix operations can be performed (e.g. multiplication, addition, transposition) efficiently (computation time negligible, linear growth of memory footprint). The results obtained through the discretized Markov chain do not deviate significantly from the exact results obtained through continuous Markov chains.

In this paper it has been shown that such an analysis is applicable to systems from a nuclear power plant. On the example of a simple water-cooling system with repairable and non-repairable components it has been illustrated that the application of ADD diagrams for analysis of the discretized Markov chain is possible. It has been shown that the probability of finding a system in one of the states after a selected period of time can be numerically calculated. The example of calculations for a period of one year at discrete intervals of one hour presented results obtained from the discrete Markov chain deviating by less than 0.3% from the exact result. It has also been shown that for a longer period of ten years with rough discretization at intervals of one year, the obtained numerical result is still close to the exact result (relative error less than 1%). The very process of conducting the analysis of the Markov chain was carried out through several steps. First of all, a continuous Markov chain is defined and it describes the dynamics of changing the state of the system. From the model defined in this way a transition rate matrix follows and can be recorded by using the ADD diagram. The normalized transition probability matrix is derived from the transition rate matrix with the assumption that the retention time in some state is determined by an exponential random variable. In a similar way, a matrix of transitional probabilities can be prepared in case of some other distribution, for example, Weibull's. Using the matrix of transitional probabilities with the power method from [11], calculations were carried out and the results obtained were presented. As indicated in the paper [4] the ADD record is suitable for the implementation of calculations of limit probabilities for Markov models that result in more than 10^{27} states (approx. a system of 90 two-state components).

Although the results obtained are promising, for future research it would be interesting to see which results will be obtained for the discretization of the Markov chain assuming the presence of other distributions or diverse distributions in the same Markov model. In addition, it is interesting to investigate the application of the discretization of Markov chains presented herein by applying ADD diagrams in the calculation of other indicators from a probabilistic analysis of multi-state systems, for example, mean time to failure (MTTF), or mean repair time (MTTR). Another interesting point is to investigate the accuracy and convergence of approaches on large systems, because the results obtained from this example suggest that this approach is robust for application on such systems. Finally, we can conclude that the application of the idea of discretization of continuous Markov chains and the application of ADD diagrams can ultimately give results good enough for practical application in the multi-state reliability analysis of systems from a nuclear power plant.

REFERENCES

[1] A. Kumar, A. Saxena, M. Ram, Multi-State Reliability Measures Analysis of Nuclear Power Plant (NPP) System, International Journal of Reliability, Quality and Safety Engineering, Vol. 27, No. 2, 2020.

[2] D. Colombo, D.T.M.P. Abreu, M.R. Martins, Application of Markov models in reliability and availability analysis: advanced topics, Safety and Reliability Modeling and its Applications, pp. 91-160, 2021.

[3] L. Xing, K.N. Fleming, and T.L. Wee, Comparison of Markov model and fault tree approach in determining initiating event frequency for systems with two train configurations, *Reliability Engineering & System Safety*, Vol. 53, No. 1, pp. 17–29, 1996.

[4] G.L. Hachtel, E. Macii, A. Pardo, and F. Somenzi, Markovian Analysis of Large Finite State Machines, *IEEE Transactions on Computer Aided Design*, Vol. 15, No. 12, pp. 1479–1493, 1996.

[5] S.B. Akers, Binary Decision Diagrams, *IEEE Transaction on Computers*, Vol. C-27, No. 6, pp. 509–516, 1978.

[6] R.M. Bryant, Graph-Based Algorithms for Boolean Function Manipulation, *IEEE Transaction on Computers*, Vol. C-35, No. 8, pp. 677–691, 1986.

[7] R. Bahar, E. Frohm, C. Gaoana, et al., Algebraic Decision Diagrams and Their Applications. *Formal Methods in System Design*, Vol. 10, No. 1, pp. 171–206, 1997.

[8] G. Saritha, M.T. Devi, and T.S.U. Maheswari, Reliability and Availability for Non-Repairable and Repairable Systems using Markov Modelling, *International Journal of Engineering Research & Technology (IJERT),* Vol. 9, No. 2, pp. 169–174, 2020.

[9] A. Pachon, F. Polito, C. Riccuti, On discrete-time semi-Markov processes, *Discrete & Continuous Dynamical Systems – Series B,* Vol. 26, No. 3, pp. 1499-1529, 2021.

[10] A. Lisnianski and G. Levitin, Multi-state System Reliability: Assessment, Optimization and Applications. World Scientific, Singapore, 2003.

[11] G.H. Golub, C.F. Van Loan, Matrix computations, 4th edition, John Hopkins University Press, 2013.

[12] F. Somenzi, I. Maidanski, Colorado University Decision Diagrams (CUDD) package, https://github.com/ivmai/cudd (accessed 3. May, 2024)